Towards Context-based Epistemic Logic

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Contents

Introduction

- **Purpose:**
- Context-based Epistemic Logic (CEL) = the logic of context-sensitive knowledge.
- (Traditional) Epistemic Logic (EL) = the logic of knowledge;

  – **CSK** is a notion abstracted from Epistemic Contextualism solution to Skepticism, which also being used to solve some other epistemic puzzles.

Introduction (cont'd)

- **Motivations**:
  - **1.** To provide a new approach to the study of **limited rationality** (LA) in Epistemic Logic.
  - Nowadays, the study of LA is extremely popular not only in the field of Epistemic Logic, but also in other fields like Game theory, Decision theory, Social software, and Artificial Intelligence (AI).

Introduction (cont'd)

- As far as I know, there are at least two approaches to deal with LA in the field of Epistemic Logic:
  - Through *awareness*: (Cf. Fagin & Halpern, 1988; & de Jager, 2009);
  - Through *access*: (Cf. Hoshi & Pacuit, 2009);
  - …

Introduction (cont'd)

– **2.** To strengthen further the connection between **epistemology** and **epistemic logic**.

– Quote:

"At first sight, the modern agenda of epistemology has little to do with logic… Now, epistemic logic started as a contribution to epistemology, or at least a tool in its modus operandi, with the seminal book ***Knowledge and Belief*** (Hintikka's, 1962,2005)."

---from (van Benthem, 2006)

Skepticism and EC

Skeptical Argument (SA)

- **Basic form** (DeRose, 1995):
  - P1: I don't know that not-*H*.
  - P2: If I don't know that not-*H*, then I don't know that *O*.
  - C: So, I don't know that *O*.

SA (cont'd)

- **Example:**
  - P1: I don't know that I am not a BIV.
  - P2: If I don't know that I am not a BIV, then I don't know that I have two hands.
  - C: So, I don't know that I have two hands.

Epistemic Contextualism (EC)

- **Quote:**

"…**EC** is the view that the proposition expressed by a given knowledge attribution ('S knows that p', 'S doesn't know that p') depends upon the context in which it is uttered."

---from (Rysiew,2009).

EC's solution to Skepticism

- The presence of P1 has changed the context, such that a higher standard of knowledge are required.

- **Advantages:** explain the persuasiveness of SA & protect the correctness of our ordinary knowledge.

Further Question

- **Philosophers:** What is a context?
  - Cf. (Barke,2004); conversational context vs. epistemic context.
- **Logicians:** How to represent a context?
  - Cf. (Stalnaker,1998); context set: a set of possible worlds (or states).

Epistemic Logic & Context Logic

Epistemic Logic ($\mathcal{EL}$)

- **Language ($\mathcal{L}_{\mathcal{E}}$):**

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid K\varphi$$

Where $p \in P$.

Note: K is the abbreviation for $K_{\boldsymbol{a}}$, since I only consider one agent $\boldsymbol{a}$ for simplicity.

$\mathcal{EL}$ (cont'd)

- **Epistemic model :**

$$\mathfrak{M} = \langle W, \approx, V \rangle$$

Where:

– W is a non-empty set;

– $\approx$ is an equivalence relation on W;

– V is a valuation mapping each $p \in P$ to a subset of W, i.e., $V(p) \subseteq \wp(W)$.

$\mathcal{EL}$ (cont'd)

- **Semantics:**

$$\mathfrak{M}, w \models p \quad \text{iff } w \in V(p);$$

$$\mathfrak{M}, w \models K\varphi \quad \text{iff for all } w' \in \approx_w, \mathfrak{M}, w' \models \varphi;$$

$$\text{where } \approx_w = \{v \mid v \in W \,\&\, w \approx v\}.$$

$\mathcal{EL}$ (cont'd)

- **Axiomatization (S5):**
  - **Taut**: All instantiations of propositional tautologies;
  - **K**: $K(\varphi \to \psi) \to (K\varphi \to K\psi)$;
  - **T**: $K\varphi \to \varphi$; (Truth)
  - **4**: $K\varphi \to KK\varphi$; (Positive introspection)
  - **5**: $\neg K\varphi \to K\neg K\varphi$; (Negative introspection)
  - **MP**: From φ and $\varphi \to \psi$, infer ψ;
  - **N**: From φ, infer $K\varphi$.

$\mathcal{EL}$ (cont'd)

- **Notation:**
  - We denote $\mathcal{EL}$-validity and $\mathcal{EL}$-provability of φ as "$\models_{\mathcal{EL}} \varphi$" and "$\vdash_{\mathcal{EL}} \varphi$", respectively.

- **Completeness of $\mathcal{EL}$:**
  - **Theorem 1**: For any $\varphi \in \mathcal{L}_{\mathcal{E}}$, $\models_{\mathcal{EL}} \varphi \Leftrightarrow \vdash_{\mathcal{EL}} \varphi$.
  - ***Proof.*** Cf. (van Ditmarsch *et al*, 2007, Chapter 7).

Context Logic ($\mathcal{CL}$)*

- **Language ($\mathcal{L}_{\mathcal{C}}$):**

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid [X]\varphi \mid [U]\varphi$$

Where $p \in P$, $X \in C$, and C is the index set of contexts.

Duals:

$\langle X \rangle \varphi = \neg [X] \neg \varphi$; $\langle U \rangle \varphi = \neg [U] \neg \varphi$.

$\mathcal{CL}$ (cont'd)

- **Context model :**

$$\mathfrak{M} = \langle W, R, V \rangle$$

Where:

- W is a non-empty set;
- R is a function mapping each $X \in C$ to a subset of W, i.e., $R_X \subseteq \wp(W)$; (*Note: R_X is short for $R(X)$.)
- V is a valuation mapping each $p \in P$ to a subset of W, i.e., $V(p) \subseteq \wp(W)$.

$\mathcal{CL}$ (cont'd)

- **Semantics:**

$$\mathfrak{M}, w \models [X]\varphi \text{ iff for all } w' \in R_X, \mathfrak{M}, w' \models \varphi;$$

$$\mathfrak{M}, w \models [U]\varphi \text{ iff for all } w' \in W, \mathfrak{M}, w' \models \varphi.$$

$\mathcal{CL}$ (cont'd)

- **Axiomatization (K45^{XY}):**
  - **Taut** plus the following, where $X, Y \in C \cup \{U\}$:
  - **K^{X}**: $[X](\varphi \to \psi) \to ([X]\varphi \to [X]\psi)$;
  - **T^{U}**: $[U]\varphi \to \varphi$;
  - **4^{XY}**:$[X]\varphi \to [Y][X]\varphi$;
  - **5^{XY}**:$\langle X \rangle\varphi \to [Y]\langle X \rangle\varphi$;
  - **MP**: From φ and $\varphi \to \psi$, infer ψ;
  - **N^{X}**: From φ, infer $[X]\varphi$.

$\mathcal{CL}$ (cont'd)

- **Notation:**
  - We denote $\mathcal{CL}$-validity and $\mathcal{CL}$-provability of φ as "$\vDash_{\mathcal{CL}} \varphi$" and "$\vdash_{\mathcal{CL}} \varphi$", respectively.

- **Completeness of $\mathcal{CL}$:**
  - **Theorem 2**: For any $\varphi \in \mathcal{L}_{\mathcal{C}}$, $\vDash_{\mathcal{CL}} \varphi \Leftrightarrow \vdash_{\mathcal{CL}} \varphi$.
  - ***Proof.*** Cf. (Grossi *et al*, 2008).

Appendix*

- **Note that**, we can easily abstract a **three-valued semantics** from context model as follows, where $s = R_X$ for some $X \in C$:
  - $\mathfrak{M}, s \Vdash p$ iff $s \subseteq V(p)$;
  - $\mathfrak{M}, s \nVdash p$ iff $s \subseteq W/V(p)$;
  - $\mathfrak{M}, s \Vdash \neg\varphi$ iff $\mathfrak{M}, s \nVdash \varphi$;
  - $\mathfrak{M}, s \nVdash \neg\varphi$ iff $\mathfrak{M}, s \Vdash \varphi$;
  - $\mathfrak{M}, s \Vdash \varphi \wedge \psi$ iff $\mathfrak{M}, s \Vdash \varphi$ **and** $\mathfrak{M}, s \Vdash \psi$;
  - $\mathfrak{M}, s \nVdash \varphi \wedge \psi$ iff $\mathfrak{M}, s \nVdash \varphi$ **or** $\mathfrak{M}, s \nVdash \psi$;
  - $\mathfrak{M}, s \Vdash \varphi \vee \psi$ iff $\mathfrak{M}, s \Vdash \varphi$ **or** $\mathfrak{M}, s \Vdash \psi$;
  - $\mathfrak{M}, s \nVdash \varphi \vee \psi$ iff $\mathfrak{M}, s \nVdash \varphi$ **and** $\mathfrak{M}, s \nVdash \psi$.

Foundations of CEL

Foundations of CEL

- **Context-based epistemic model:**

$$\mathfrak{M} = \langle W, R, \approx, V \rangle$$

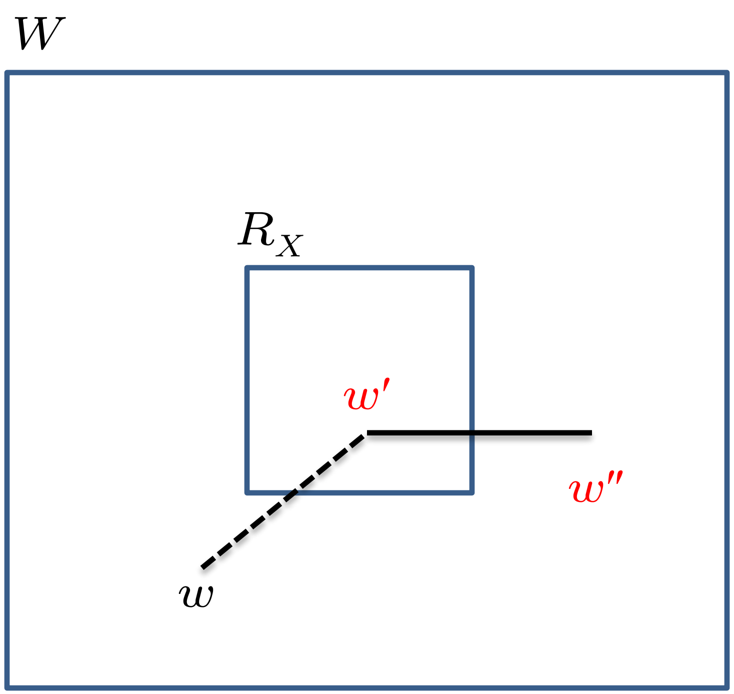
Where:

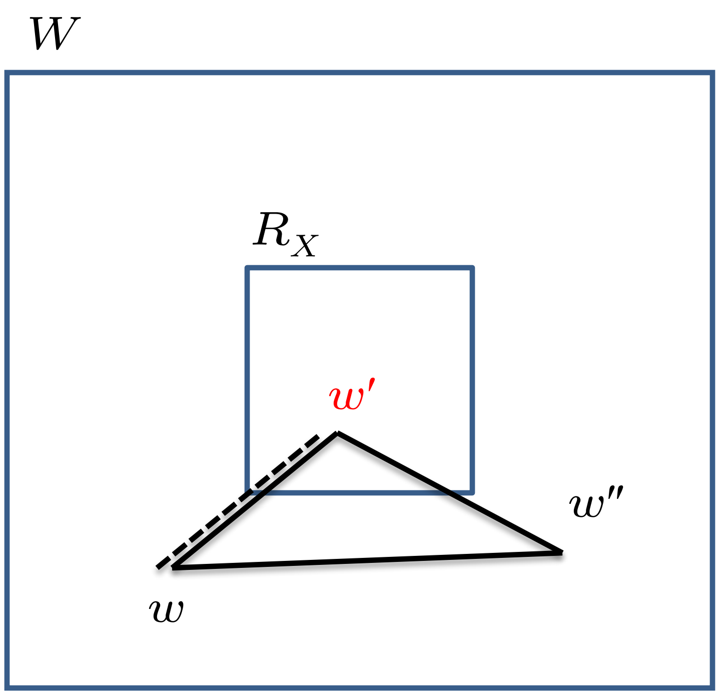
- W is a non-empty set;
- R is a function mapping each $X \in C$ to a subset of W, i.e., $R_X \subseteq \wp(W)$;
- $\approx$ is an equivalence relation on W;
- V is a valuation mapping each $p \in P$ to a subset of W, i.e., $V(p) \subseteq \wp(W)$.

Foundations of CEL (cont'd)

- **Definitions of CSK:**
  - **Static-style:**
    - $\mathfrak{M}, w \vDash [X]K\varphi$ iff for all $w' \in R_X$, $\mathfrak{M}, w' \vDash K\varphi$
      iff for all $w'' \in \approx_{w'}$, $\mathfrak{M}, w'' \vDash \varphi$.
  - **Dynamic-style:**
    - $\mathfrak{M}, w \vDash K^X\varphi$ iff for all $w' \in \approx_w \cap R_X$, $\mathfrak{M}, w' \vDash \varphi$.
    - Similarly, we can define K^U, which is reduced to the standard epistemic operator K.

Comparisons





Static: $w \models [X]K\varphi$, iff $w' \models \varphi$ and $w'' \models \varphi$.

Dynamic: $w \models K^X\varphi$ iff $w' \models \varphi$.

Figure 1. Definitions of CSK: Static vs. Dynamic

Comparisons (cont'd)

- **For static:**
  - (i) CSK is defined in terms of context-free knowledge;* (which seems to be highly questionable.)
  - (ii) $K\varphi \rightarrow [X]K\varphi$ is not valid;
  - (iii) Nonetheless, $[X]K\varphi \rightarrow [X]\varphi$ is valid.
- **For dynamic:**
  - (i) CSK is defined independently;
  - (ii) $K\varphi \rightarrow K^X\varphi$ is valid;
  - (iii) Nonetheless, $K^X\varphi \rightarrow [X]\varphi$ is not valid.

Comparisons (cont'd)

– Further,

– **Static-style** definition seems to be consistent with an **objective** understanding of context.* (However, I haven't come up with any concrete example yet.)

– **Dynamic-style** definition seems to work well with the **subjective** understanding of context (esp. **common assumptions**, cf. example below).

SA in Dynamic CSK

- **For example:**

  p: I am not a BIV;

  q: I feel that I have two hands;

  r: I have to hands.

  The model is as indicated in **Figure 2**, where:

  $V(p)=\{u,v\}$;

  $V(q)=\{u,s\}$;

  $V(r)=\{u\}$.

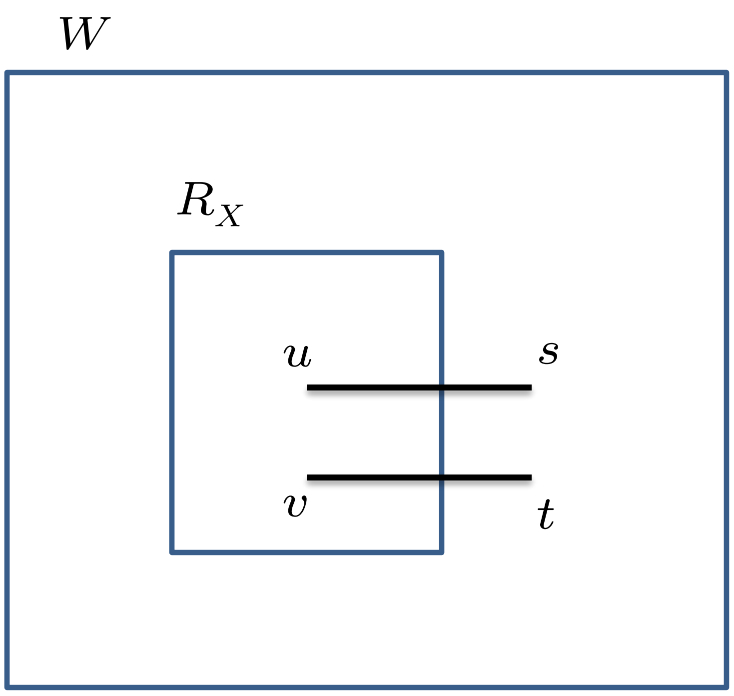


Figure 2. SA in Dynamic CSK

SA in Dynamic CSK (cont'd)

(i) In all states, I know my feelings (i.e., either I feel that I have two hands or not).

(ii) If u is the real state, then with the assumption of X, I know that I have two hands and I know that I am not a BIV.

(iii) When P1 of SA appears, the context has been extended to contain either s or t (depending on whether u or v is taken as the actual state).

(iv) So, the extension of context set corresponds to the retraction of assumption. *(In next paper, I will revisit this example with more details after I work out the whole update mechanics.)

Candidates: *CELs* & *CELd*

$\mathcal{CEL}s$

- **Language ($\mathcal{L}_{\mathcal{CE}s}$):**

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid K\varphi \mid [X]\varphi \mid [U]\varphi$$

Where $p \in P$, $X \in C$, and C is the index set of contexts.

- **Semantics:**

$\mathfrak{M}, w \vDash K\varphi$ iff for all $w' \in \approx_w$, $\mathfrak{M}, w' \vDash \varphi$;

$\mathfrak{M}, w \vDash [X]\varphi$ iff for all $w' \in R_X$, $\mathfrak{M}, w' \vDash \varphi$;

$\mathfrak{M}, w \vDash [U]\varphi$ iff for all $w' \in W$, $\mathfrak{M}, w' \vDash \varphi$.

$\mathcal{CEL}s$ (cont'd)

- **Axiomatization:**
  - All axioms and rules of $\mathcal{EL}$ and $\mathcal{CL}$, plus the axiom schemas below:
  - $\mathbf{4}^{XK}$: $[X]\varphi \to K[X]\varphi$;
  - $\mathbf{5}^{XK}$: $\langle X\rangle\varphi \to K\langle X\rangle\varphi$.

  ***Remark:** Knowledge operator behaves just half-like context, since the following schemas are generally invalid:

    - $\mathbf{4}^{KX}$: $K\varphi \to [X]K\varphi$;
    - $\mathbf{5}^{KX}$: $\langle K\rangle\varphi \to [X]\langle K\rangle\varphi$.

$\mathcal{CEL}s$ (cont'd)

- **Notation:**
  - We denote $\mathcal{CEL}s$-validity and $\mathcal{CEL}s$-provability of φ as "$\vDash_{\mathcal{CEL}s}\varphi$" and "$\vdash_{\mathcal{CEL}s}\varphi$", respectively.

- **Completeness of $\mathcal{CEL}s$:**
  - **Theorem 3**: For any $\varphi\in\mathcal{L}_{\mathcal{CE}s}$, $\vDash_{\mathcal{CEL}s}\varphi \Leftrightarrow \vdash_{\mathcal{CEL}s}\varphi$.
  - ***Proof.*** Cf. (Xu, forthcoming).

$\mathcal{CEL}d$

- **Language ($\mathcal{L}_{\mathcal{CE}s}$):**

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi \mid K\varphi \mid [X]\varphi \mid [U]\varphi \mid K^X\varphi$$

Where $p \in P$, $X \in C$, and C is the index set of contexts.

- **Semantics:**

$\mathfrak{M}, w \vDash K^X\varphi$ iff for all $w' \in \approx_w \cap R_X$, $\mathfrak{M}, w' \vDash \varphi$.

$\mathcal{CEL}d$ (cont'd)

- **Axiomatization:**
  - All axioms and rules of $\mathcal{CEL}s$, plus the axiom schemas and rule below:
  - **K**^{KX}: $K^X(\varphi \to \psi) \to (K^X\varphi \to K^X\psi)$;
  - $K \cap X$: $K\varphi \vee [X]\varphi \to K^X\varphi$;
  - **N**^{KX}: From φ, infer $K^X\varphi$.

$\mathcal{CEL}d$ (cont'd)

- **Notation:**
  - We denote $\mathcal{CEL}d$-validity and $\mathcal{CEL}d$-provability of φ as "$\vDash_{\mathcal{CEL}d}\varphi$" and "$\vdash_{\mathcal{CEL}d}\varphi$", respectively.

- **Completeness of $\mathcal{CEL}d$:**
  - **Theorem 4**: For any $\varphi\in\mathcal{L}_{\mathcal{CE}d}$, $\vDash_{\mathcal{CEL}d}\varphi \Leftrightarrow \vdash_{\mathcal{CEL}d}\varphi$.
  - ***Proof.*** Cf. (Xu, forthcoming).

Conclusion

– I have introduced the philosophical background of CSK and preliminaries: $\mathcal{EL}$ & $\mathcal{CL}$;

– After that, I have proposed two distinct ways of defining CSK and made some detailed contrast;

– Further, I have obtained two candidate systems of CEL (namely, $\mathcal{CEL}s$ & $\mathcal{CEL}d$) and proved their completeness.

Future Work

- **Future work:**
  - Develop the update mechanics of $\mathcal{CEL}s$ & $\mathcal{CEL}d$;
    - (Cf., Dynamic Context Logic, Aucher et al, 2009).
  - Compare with update semantics;
    - (Cf., Veltman, 1996; & de Jager, 2009).
  - Extend with more philosophical discussion;
    - (Cf. Lewis, 1996).
  - Default reasoning?…

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Thank you!